Available online at www.sciencedirect.com
SCIENCE DIRECT®

ACADEMIC PRESS

Journal of Sound and Vibration 261 (2003) 557–566

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Letter to the Editor

Frequencies of oscillators with fractional-power non-linearities

H.P.W. Gottlieb*

School of Science, Griffith University, Nathan, Queensland 4111, Australia

Received 22 April 2002; accepted 30 May 2002

1. Introduction

In a recent paper, Mickens [1] drew attention to the idea that the method of harmonic balance (HB) could also easily be applied to oscillators with a single term non-linearity involving an inverse odd-integer power by the simple expedient of raising each side of the force equation to that power, so that in the rewritten equation the acceleration term was raised to the integer power and the coefficients of the lowest order harmonic could then immediately be matched. One purpose of this paper is to give both the exact expression for the frequency of oscillators with general, positive power non-linearity in the force equation and the corresponding direct first order HB approximate result utilizing the first Fourier coefficient, against which more approximate HB results such as those derived in Ref. [1] may be compared. This paper then explores the effects on the accuracy of the HB approach as successive manipulations of the underlying acceleration equation, such as that made in Ref. [1], which apparently make the HB method easier to apply, are carried out before balancing is effected. Explicit results are given for the case of inverse odd and inverse even integer powers, as well as for more general rational powers. Limiting cases as the power itself tends to zero are also investigated. Exact solutions for some examples are verified by numerical integration of the differential equations (d.e.'s), utilizing the software ODE Workbench [2]. Attention here is restricted primarily to rational powers less than unity in the non-linearity in the acceleration equation. Some related material may be elicited from Gelb and Vander Velde [3], but their emphasis was on positive integer powers, and their approach was in the somewhat unfamiliar context of input–output describing functions.

The method of HB has been used for many instances of oscillators [4,5] and even in other fields such as the non-linear Klein–Gordon equation [6]. An investigation of the various ways in which it may be utilized is therefore in order. It will be found here that, the more preliminary manipulations are made on the original d.e. before proceeding, the easier the application of the HB method becomes, but the less accurate the results appear.

*Tel.: +61-7-3875-7556; fax: +61-7-3875-7656.
E-mail address: h.gottlieb@sct.gu.edu.au (H.P.W. Gottlieb).

PII: S0022-460X(02)01003-9

It is first of all noted that Eq. (7b) of Ref. [1], viz.

$$\frac{d^2x}{dt^2} + x^{1/(2n+1)} = 0 \tag{1.1a}$$

is to be read in conjunction with the second equation of (5) in Ref. [1], which states that the "force function" (second term in Eq. (1.1a) above) is understood to be an odd function of x, changing sign when x changes sign. It is therefore perhaps more direct to include this idea into the original equation, and rewrite Eq. (1.1a) as

$$\frac{d^2x}{dt^2} + \mathrm{sgn}(x)|x|^{1/(2n+1)} = 0, \tag{1.1b}$$

where sgn(x) is the sign function, equal to $+1$ if $x > 0$, 0 if $x = 0$, and -1 if $x < 0$. Indeed, the desirability of this stance can be appreciated by taking the limit as $n \to \infty$. Eq. (1.1a) as it stands appears to approach $\ddot{x} + 1 = 0$, which does not have periodic solutions. On the other hand, Eq. (1.1b) approaches $\ddot{x} + \mathrm{sgn}(x) = 0$, which does have periodic solutions, as investigated by Lipscomb and Mickens [7], and against which the behaviour of solutions of Eq. (1.1b) can be compared as n becomes large. This will be elaborated upon later.

Thus modified, the particular case Eq. (8) in Ref. [1] becomes

$$\frac{d^2x}{dt^2} + \mathrm{sgn}(x)|x|^{1/3} = 0. \tag{1.2}$$

This may be integrated once to yield an "energy equation"

$$\dot{x}^2 + (\tfrac{3}{2})(x^4)^{1/3} = (\tfrac{3}{2})A^{4/3}, \tag{1.3}$$

where the initial conditions are

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{1.4}$$

Since both terms on the left-hand side of Eq. (1.3) are always positive, these phase plane curves are closed, and the motion is periodic. (A more formal proof of periodicity is given in Ref. [1].)

The period T for oscillations is then given by

$$T = 4\int_0^A \frac{dx}{|\dot{x}|}. \tag{1.5}$$

The successive substitutions $x = Ay$ and $z = y^{4/3}$ enable integral (1.5) to be expressed in terms of a Beta function and thence in terms of Gamma functions. (For general basic mathematical formulae and relations, see, e.g., Ref. [8, Chapter 11]). There results the explicit exact (EX) expression for the period of this power $\frac{1}{3}$ force law

$$T_{1/3}^{EX} = 4\sqrt{6}\sqrt{\pi}[\Gamma(3/4)/\Gamma(1/4)]A^{1/3}. \tag{1.6}$$

The numerical value of the quarter-period obtained from expression (1.6) is 1.46741610770 . This is quoted here to so many significant figures as a check on this mathematics, or, conversely, as a check on the accuracy of the numerical software [2]: numerical integration of the oscillator equation (1.2) with initial conditions (1.4) and $A = 1$, and with judicious choice of settings, gave agreement to 11 significant figures. In all the numerical examples to follow for the exact results, similar agreement, to about 9 or 10 significant figures, was attained (although so many places will not be quoted).

The angular frequency $\Omega \equiv 2\pi/T$ corresponding to the exact result (1.6) has value $1.070451/A^{1/3}$. By way of comparison, the HB approximation for Ω obtained in Ref. [1] is $(\frac{4}{3})^{1/6}/A^{1/3}$. (Note that the dependence on the initial amplitude is the same.) The numerical value of the constant in this HB result is 1.049115. The HB method as used in Ref. [1] therefore underestimates the frequency, with a relative error of less than 2%.

Mickens [1] gave the HB approximation formula for the angular frequency for arbitrary inverse odd-integer powers for Eq. (1.1). In the following, we extend both exact and HB results to more general fractional-power equations.

2. Exact results for general positive powers

The oscillator equation with single-term positive-power non-linearity

$$\ddot{x} = -\mathrm{sgn}(x)|x|^p \tag{2.1}$$

(together with initial conditions (1.4)) corresponds to the energy equation

$$\dot{x}^2 = \frac{2}{p+1}[A^{p+1} - |x|^{p+1}]. \tag{2.2}$$

By following a procedure similar to that described in the Introduction, it is possible to express the exact angular frequency of solutions to Eq. (2.1) in terms of Gamma functions, here given in a form suitable for positive powers $p<1$. (In deriving this result, a recurrence relation [8, Chapter 11] for the Gamma function in the numerator has been used so that its argument is less than 1.)

$$\Omega_p^{EX} = \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{(1-p)}{\sqrt{p+1}} \frac{\Gamma\left(\dfrac{1-p}{2(p+1)}\right)}{\Gamma\left(\dfrac{1}{p+1}\right)} 1/A^{(1-p/2)}. \tag{2.3}$$

(Gelb and Vander Velde [3, p. 166], only for integer values of p, found an expression in terms of an integral, but apparently did not proceed to re-express it in terms of Beta or Gamma functions.) Some exact results for a range of rational values of $p<1$ are included in Table 1. As p decreases, the frequency values slowly increase.

Note that, as $p \to 0$, this tends (since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$) to

$$\Omega_{p\to 0}^{EX} = (\pi/\sqrt{8})/A^{(1/2)} = 1.110721/\sqrt{A}. \tag{2.4}$$

As indicated in the Introduction, the limiting oscillator equation as $p \to 0$ is

$$\ddot{x} + \mathrm{sgn}(x) = 0. \tag{2.5}$$

The solution is obtained explicitly (cf., Refs. [7,9]) for the quarter-period $T/4$, subject to the initial conditions (1.4)

$$x = A - (1/2)t^2; \quad 0 \leqslant t \leqslant T/4. \tag{2.6}$$

Table 1
Comparison of some values of the angular frequency Ω of solutions to the non-linear oscillator equation (2.1) for various rational powers p ($0 \leqslant p \leqslant 0.1$)

p	α	Exact Ω^{EX}	Harmonic balance		
			Fourier Ω^{HBF}	Powering once Ω^{HBP} (Error)	Further squaring Ω^{HBPS}(Error)
1	0	1	1		
3/4	−1/8	1.024957	1.025674	1.015593 (−1%)	1.008381 (−2%)
5/7	−1/7	1.028660	1.029613	1.009584 (−2%)	
2/3	−1/6	1.033652	1.034982	1.020845 (−1%)	1.015309 (−2%)
3/5	−1/5	1.040749	1.042734	1.018399 (−2%)	
1/2	−1/4	1.051637	1.054910	1.062252 (+1%)	1.036615 (−1%)
3/7	−2/7	1.059596	1.064045	1.022817 (−3%)	
1/3	−1/3	1.070451	1.076845	1.049115[1] (−2%)	
1/4	−3/8	1.080181	1.088681	1.068393 (−1%)	1.038441 (−4%)
1/5	−2/5	1.086126	1.096092	1.048122 (−3%)	
1/6	−5/12	1.090133	1.101695	1.061091 (−3%)	1.033719 (−5%)
1/7	−3/7	1.093018	1.104867	1.044052 (−4%)	
1/8	−7/16	1.095194	1.107679	1.054239 (−4%)	1.029635 (−6%)
1/9	−4/9	1.096894	1.109890	1.040169 (−5%)	
1/10	−9/20	1.098258	1.111675	1.048670 (−5%)	1.026418 (−7%)
1/11	−5/11	1.099377	1.113145	1.036840 (−6%)	
—					
1/∞	−1/2	1.110721	1.128379	1 (−10%)	
0	−1/2	1.110721	1.128379		

The constant tabulated is that appearing in the relation Ω = CONSTANT. A^{α} where $\alpha = -(1-p)/2$. Eq. (2.3) gives the exact results. Eqs. (3.4), (4.3), (4.6), (4.9), (4.12), (4.14), (5.1) give various first order harmonic balance approximations. The last two rows in the Table correspond to the limit of the solution for inverse integer powers, and the solution of the limit equation, respectively.

When $x = 0$, $t = T/4$, whence the frequency $\Omega \equiv 2\pi/T$ of oscillations is

$$\Omega^{EX}_{p=0} = (\pi/\sqrt{8})/A^{(1/2)} = 1.110721/\sqrt{A}. \tag{2.7}$$

Comparison with (2.4) confirms consistency: the limit value of the period of the exact solution to the power p oscillator d.e. as p approaches zero agrees with the period of the exact solution to the limit d.e. Physically, the two cases correspond to the limit of a power potential as its power decreases to unity, and a V-shaped potential, respectively.

3. Direct harmonic balance via first Fourier coefficient

The direct first order HB approach [5] assumes an approximate solution to the original oscillator d.e. (2.1) as it stands, of the form

$$x = A\cos\Omega t, \tag{3.1}$$

so $\ddot{x} = -\Omega^2 A\cos\Omega t$, and "balances" the lowest order harmonic across the relevant equation. Substitution into the original Eq. (2.1) and immediate application of the HB method requires the evaluation of the first Fourier (cosine) coefficient of $\mathrm{sgn}(\cos\Omega t)|\cos\Omega t|^p$. This is given by the integral $a_1 = (4/\pi)I_{p+1}$, where

$$I_s = \int_0^{\pi/2} \cos^s\theta\,\mathrm{d}\theta. \tag{3.2}$$

The angular frequency corresponding to power p using this "Harmonic Balance Fourier" (HBF) approach is then

$$\Omega_p^{HBF} = \frac{2}{\sqrt{\pi}}\sqrt{I_{p+1}}/A^{(1-p)/2}. \tag{3.3}$$

Upon use of a number of changes of variables, integral (3.2) may be expressed as a Beta function [10, p. 412] and thence in terms of Gamma functions. The result is

$$\Omega_p^{HBF} = \frac{2^{(p+2)/2}}{\sqrt{\pi}}\frac{\Gamma((p+2)/2)}{\sqrt{\Gamma(p+2)}}1/A^{(1-p)/2}. \tag{3.4}$$

(Gelb and Vander Velde [3, p. 166], but only for positive integer values of p, give a slightly less convenient form; equivalence (for integer p) to result (3.4) may be proved using the duplication formula for Gamma functions [8].)

For $p = \frac{1}{3}$, the numerical factor in Eq. (3.4) has value 1.076845, an error of only 0.6%, which is less than the error, mentioned in the Introduction above, obtained in Ref. [1], which manipulated the original d.e. first, resulting in an easier application of HB. Further numerical results for HBF are given in the HB "Fourier" column in Table 1.

Note that, as $p \to 0$, Eq. (3.4) tends (since $\Gamma(n) = (n-1)!$ for positive integer n) to

$$\Omega_{p\to 0}^{HBF} = (2/\sqrt{\pi})/A^{(1/2)} = 1.128379/\sqrt{A}. \tag{3.5}$$

Upon comparison with Eq. (2.4), this shows that, as $p \to 0$, the limit of the Harmonic Balance Fourier result does not equal the limit of the exact result, with the numerical factor being overestimated, albeit by less than 2%, although the amplitude dependence is correct. These results will be compared with other method results for the limiting d.e. later.

The HB solution to the limit oscillator equation (2.5) with Eq. (3.1) may be found by noting that in the Fourier series for $\mathrm{sgn}(\cos\theta)$ the first non-zero coefficient, that of $\cos\theta$, is $4/\pi$. The first order HB angular frequency approximation to the limit Eq. (2.5) is therefore given by

$$\Omega_{p=0}^{HBF} = (2/\sqrt{\pi})/A^{1/2} \tag{3.6}$$

agreeing with (3.5). Thus the limit of the HBF method period of the power p d.e. as p approaches zero is equal to the HBF method period of the limit d.e., but this differs somewhat from the exact result given in Section 2 above.

4. Harmonic balance after manipulation of the underlying oscillator equation

4.1. Inverse integer powers

4.1.1. Inverse odd integers $p = 1/(2n+1)$

For inverse odd integers $p = 1/(2n+1)$; $n = 1, 2, 3, \ldots$; Mickens [1] first raised both sides of Eq. (2.1) to the $(2n+1)$th power and then "balanced" the coefficients of the $\cos \Omega t$ term (the modulus sign dropping out). Use was made of the identity (cf. [8, Section 2.4.1])

$$\cos^{2n+1}\theta = \frac{1}{2^{2n}}\binom{2n+1}{n}\cos\theta + \text{higher order harmonics}, \tag{4.1}$$

where for positive integers N and $M < N$,

$$\binom{N}{M} \equiv \frac{N!}{(N-M)!M!} \tag{4.2}$$

and higher order harmonics are not considered. It is important to note that Eq. (4.1) is a finite sum, with fairly simple coefficients for low n (see e.g., the standard identities in Ref. [10, pp. 32–33]), and the first coefficient may simply be read off. For future reference, we record the Mickens result (Eq. (24) in Ref. [1]) for the manipulated "powered" (P) d.e., written here in terms of factorials and now labelling the frequency by the actual power in the oscillator d.e. (2.1)

$$\Omega^{HBP}_{1/(2n+1)} = 2^{\left(\frac{n}{2n+1}\right)}\left[\frac{(n+1)!n!}{(2n+1)!}\right]^{\frac{1}{2(2n+1)}} 1/A^{\left(\frac{n}{2n+1}\right)}. \tag{4.3}$$

For example, for $n = 2$ ($p = \frac{1}{5}$), this gives $(\frac{8}{5})^{(1/10)}/A^{2/5}$. Numerical values for several cases are given in the HB "Powering once" column in Table 1. They are less accurate than values using the HB Fourier approximate approach of Section 3 above. The ease of reading off coefficients in Eq. (4.1) relevant to the manipulated equation, as against the more complicated evaluation of the first Fourier coefficient integral relevant to the original unmanipulated equation, has to be offset against a sacrifice of accuracy.

4.1.2. Inverse even integers $p = 1/(2n)$

For the HB method in the manner of Mickens [1], as summarized in Section 4.1.1. above for the inverse-odd-integer case, the inverse-even-integer case requires some slight modification.

First of all, there is the trigonometric identity (finite sum) which has a different form [8, Section 2.4.1]

$$\cos^{2n}\theta = \frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2} + \text{even harmonic terms} \tag{4.4}$$

for $n = 1,2,3,\ldots$. Secondly, when the $(2n)$th power is taken, there is a modulus remaining on the right-hand side

$$(\ddot{x})^{2n} = |x|. \tag{4.5}$$

The first term in Eq. (4.4) is easily used in the left-hand side of Eq. (4.5) for first order HB. However, in the right-hand side of Eq. (4.5), one still needs to find the constant term in the

Fourier cosine series of the function $|\cos \Omega t|$. This can be seen to be $(\frac{1}{2})(2/\pi)2\int_0^{\pi/2}\cos\theta\,\mathrm{d}\theta = 2/\pi$. Balancing constant terms in the manipulated d.e. (4.5) then yields the "Harmonic Balance Powering" (HBP) formula

$$\Omega^{HBP}_{1/(2n)} = \sqrt{2}\left(\frac{2}{\pi}\right)^{\left(\frac{1}{4n}\right)}\left[\frac{(n!)^2}{(2n)!}\right]^{1/(4n)} 1/A^{\left(\frac{2n-1}{4n}\right)}. \tag{4.6}$$

For example, for $n=1$ $(p=\frac{1}{2})$, this gives $(4/\pi)^{(1/4)}/A^{1/4}$. For $n=2$ $(p=\frac{1}{4})$, there results $[16/(3\pi)]^{(1/8)}/A^{3/8}$. Some numerical results for this HBP procedure are compared with the exact and HBF results for several values of $2n$ in Table 1.

4.1.3. Results as n tends to infinity

It is again interesting to investigate limits as $n\to\infty$. To determine the limit as $n\to\infty$ of, for instance, the HB expression (4.3) for the power law oscillator with inverse odd integer power, one needs to invoke Stirling's formula [8, Chapter 11] $J!\sim(J/\mathrm{e})^J\sqrt{(2\pi J)}$ for integer J, and the fact that $J^{(1/J)}\to 1$ as $J\to\infty$. The result is

$$\Omega^{HBP}_{1/(2n+1)} \to 1/A^{(1/2)}. \tag{4.7}$$

Although this again has the same amplitude dependence as the exact result (2.7), the numerical coefficient differs from that exact result, and underestimates the frequency now by about 10%; it also differs from Eq. (3.6), which overestimated the frequency only slightly. Thus the limit value of the frequency of the first order manipulated HB solution to the reciprocal odd integer power law oscillator does not agree with the frequency of the HB solution to the limit differential equation; and neither agrees with the exact solution. This highlights some problems which may be associated with the HB approaches. It can similarly be shown that, in the limit as $n\to\infty$, the approximate solution (4.6) for inverse even integer powers also tends to the value on the right-hand side of Eq. (4.7), so the HB result in this even case exhibits the same inconsistency as the odd case above.

4.2. Powers which are ratios of odd integers

We next investigate the case when the power of the non-linearity term in the oscillator d.e. is a fraction expressible as the ratio of (co-prime) odd integers

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\mathrm{sgn}(x)|x|^{(2m+1/2n+1)}, \quad m,n=1,2,3,\ldots;\ m\neq n. \tag{4.8}$$

(The case $m=n$ is just the ordinary linear oscillator, with $\Omega=1$ ($T=2\pi$.) The case $m=0$, $n\geqslant 1$ corresponds to the paper by Mickens [1]; and see Section 4.1.1 above. The case $n=0$, $m\geqslant 1$, including the case of large m, was investigated some time ago by the author [11]; see also Nayfeh and Mook [4, p. 92]).

The HB approach may be applied to the manipulated d.e. Thus both sides of Eq. (4.8) are raised to the $(2n+1)$th power, so that both sides contain odd powers. Eqs. (3.1) and (4.1) are used to obtain easily the first order HB result

$$\Omega^{HBP}_{(2m+1)/(2n+1)} = 2^{\left(\frac{n-m}{2n+1}\right)}\left[\frac{(2m+1)!(n+1)!n!}{(2n+1)!(m+1)!m!}\right]^{\left(\frac{1}{2(2n+1)}\right)} 1/A^{\left(\frac{n-m}{2n+1}\right)}. \tag{4.9}$$

Some numerical values are given in the HB Powering once column in Table 1, for comparison with exact and HBF results.

As an example of the necessity for cancelling factors in the power, one numerical example is mentioned. For $m = 1$, $n = 4$, i.e., power retained as $\frac{3}{9}$, the HB approximate angular frequency (for $A = 1$) by Eq. (4.9) is 1.024. For $m = 0$, $n = 1$, i.e., power $\frac{1}{3}$ used, it is 1.049. The exact value is 1.070, and is better approximated by the simplified fraction power.

4.3. Powers which are ratios involving an even integer

This section discusses oscillator equations of the form

$$\ddot{x} = -\mathrm{sgn}(x)|x|^{(\mu/\nu)}, \tag{4.10}$$

where μ and ν are co-prime positive integers, one of which is odd and the other even (they cannot both be even). The amplitude dependence of Ω is $1/A^{(\nu-\mu)/(2\nu)}$. (For $p<1$, $\mu<\nu$.) The HB approach to solving the oscillator equation (4.10) in the spirit of Ref. [1] starts off by manipulating it by raising each side to the νth power. Then the formulae depend on whether μ or ν is the even integer.

4.3.1. Powers of the form $(2m+1)/(2n)$

Here, for m, n =1,2,3,…, the manipulated d.e. takes the form

$$(\ddot{x})^{2n} = |x|^{2m+1}. \tag{4.11}$$

In the HB approach, for the left-hand side the coefficient ‘‘reading’’ formula (4.4) is used. For the right-hand side, the constant term in the Fourier cosine series for $|\cos\theta|^{2m+1}$ is required. After use has been made of a formula in Ref. [8, p. 246], the eventual result is found to be

$$\Omega^{HBP}_{(2m+1)/(2n)} = \sqrt{2}\left(\frac{2}{\pi}\right)^{\left(\frac{1}{4n}\right)}\left[\frac{(2m)!!(n!)^2}{(2m+1)!!(2n)!}\right]^{\frac{1}{4n}} 1/A^{\left(\frac{2n-2m-1}{4n}\right)}, \tag{4.12}$$

where the double factorials have factors decreasing in steps of 2 down to 2 or 1. The power of the amplitude A in the denominator is positive if the power of the oscillator non-linearity is less than unity. For example, for $m = 1$ and $n = 2$ (power $\frac{3}{4}$), this gives $\sqrt{2}\,[2/(9\pi)]^{(1/8)}/A^{1/8}$.

4.3.2. Powers of the form $(2m)/(2n+1)$

Here, for m, n =1,2,3,…, the manipulated d.e. is

$$(\ddot{x})^{2n+1} = -\mathrm{sgn}(x)x^{2m}. \tag{4.13}$$

In the HB approach, for the left-hand side the coefficient reading formula (4.1) is used. For the right-hand side, the first harmonic term in the Fourier cosine series for $\mathrm{sgn}(\cos\theta)\cos^{2m}\theta$ is required. This is found to involve the same integral as occurs in Section 4.3.1. The eventual result is

$$\Omega^{HBP}_{(2m)/(2n+1)} = \sqrt{2}\left(\frac{2}{\pi}\right)^{\left(\frac{1}{2(2n+1)}\right)}\left[\frac{(2m)!!(n+1)!n!}{(2m+1)!!(2n+1)!}\right]^{1/2(2n+1)} 1/A^{\left(\frac{2n-2m+1}{2(2n+1)}\right)}. \tag{4.14}$$

For example, for $m = 1$ and $n = 1$ (power $\frac{2}{3}$), this gives $[(32)/(9\pi)]^{(1/6)}/A^{1/6}$.

All rational powers have now been covered.

5. Further manipulation of the underlying oscillator equation

Where the manipulated oscillator equation still involved an odd integer power on one side and an even integer power on the other (they cannot both be even, after the original rational power has been simplified), it was seen above that a Fourier coefficient still had to be evaluated by integration, rather than just reading off the first coefficient from the finite sum (4.1) or (4.4). It is tempting in this case to manipulate the d.e. still further, by squaring (S) both sides, so that in the subsequent HB approximation to this doubly raised d.e. the constant terms on either side may simply be read off from the first term in identity (4.4).

This section reports on the results of this procedure, starting with the oscillator equation (4.10) wherein one of μ, ν is even. In the manner just described, after first manipulating the original oscillator d.e. by raising both sides to the νth power and then further squaring both sides, one obtains

$$\Omega^{HBPS}_{\mu/\nu} = 2^{\left(\frac{\nu-\mu}{2\nu}\right)}\left[\frac{(2\mu)!(\nu!)^2}{(2\nu)!(\mu!)^2}\right]^{1/(4\nu)} 1/A^{\left(\frac{\nu-\mu}{2\nu}\right)}. \tag{5.1}$$

Formula (5.1) also embraces the case $\mu = 1$, $\nu = 2n$, i.e., the result following from the further squaring of both sides of the d.e. (4.5). Some numerical values obtained from Eq. (5.1) are given in the HB "Further Squaring" column in Table 1. They are seen to lead to greater errors than those in the previous column, which only powered the d.e. once.

6. Discussion and conclusion

One feature in favour of the first order HB methods for oscillator equations of the type considered in this paper is that they always predict the correct dependence on the displacement amplitude A. However, it should be appreciated that this dependence may also be deduced from a kind of dimensional analysis since the equations only contain one non-linear term. Specifically, Eq. (2.1) may be written as

$$\mathrm{d}^2[x/A]/\mathrm{d}[A^{(p-1)/2}t]^2 \sim (x/A)^p, \tag{6.1}$$

whence

$$\Omega \sim A^{(p-1)/2}. \tag{6.2}$$

From Table 1, it can be seen that as the rational power p decreases from 1 to 0, the exact angular frequency of oscillation slowly but steadily increases to the limit value. The Harmonic Balance Fourier value also steadily increases, but to a slightly larger (by about 2%) limiting value. On the other hand, the table shows that, after manipulation(s) of the underlying d.e. to simplify the use of the first order HB process, the approximate frequencies exhibit somewhat erratic behaviour as p decreases.

For the rational powers analyzed, the HB approximations always appear to estimate the frequency with an error of less than about 10%, so they could all be considered fairly successful. However, since the exact values always lie between 1 and about 1.11, one could ask the question: Is the HB result any better than the crude guess of ignoring the fact that $p<1$ and just taking the linear oscillator ($p=1$) value $\Omega=1$ as an approximation? To within 10%, this crude value is a reasonable approximation. However, Table 1 shows that the HB results are better than this, especially for $p\geqslant\frac{1}{2}$, and especially following less manipulation of the oscillator d.e. itself.

It therefore seems that a tacit principle of the HB approach, that the d.e. is subject to the least manipulation before application, is well warranted. However, this better accuracy must be offset against the comparative complications in evaluating coefficients, as compared with the easier reading off of coefficients after manipulations of the d.e., which do subsequently lead to less accurate approximations.

Acknowledgements

The author thanks the Department of Mathematics at the University of Queensland for hospitality during an Academic Studies Program visit, when most of this work was carried out.

References

[1] R.E. Mickens, Oscillations in an $x^{4/3}$ potential, Journal of Sound and Vibration 246 (2001) 375–378.
[2] J.M. Aguirregabiria, ODE Workbench, American Institute of Physics, New York, 1994 (PC Version 1.50).
[3] A. Gelb, W.E. Vander Velde, Multiple-input Describing Functions and Nonlinear System Design, McGraw-Hill, New York, 1968.
[4] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations, Wiley, New York, 1979.
[5] R.E. Mickens, Oscillations in Planar Dynamic Systems, World Scientific, Singapore, 1996.
[6] C.W. Lim, B.S. Wu, L.H. He, A new approximate analytical approach for dispersion relation of the nonlinear Klein–Gordon equation, Chaos 11 (2001) 843–848.
[7] T. Lipscomb, R.E. Mickens, Exact solution to the antisymmetric, constant force oscillator equation, Journal of Sound and Vibration 169 (1994) 138–140.
[8] A. Jeffrey, Handbook of Mathematical Formulas and Integrals, Academic Press, San Diego, 1995.
[9] P.K.G. Potti, M.S. Sarma, B.N. Rao, On the exact periodic solution for $\ddot{x}+\text{sign}(x)=0$, Journal of Sound and Vibration 220 (1999) 380–383.
[10] I.S. Gradshteyn, I.M. Ryzhik (A. Jeffrey (Ed.)), Table of Integrals, Series, and Products, 5th Edition, Academic Press, London, 1994.
[11] H.P.W. Gottlieb, On vibrations of a string with a mid-point mass, and related non-linear equations, Journal of Sound and Vibration 155 (1992) 382–384 (Errata: Journal of Sound and Vibration 159 (1992) 571: although the headline on the paper itself reads Vol. 154 (number 3), this paper was actually published in Vol. 155 (number 2).)